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A note on the minimum size of a vertex pancyclic graph

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Abstract

Let G be a graph on $n \geqslant 3$ vertices. Then G is vertex pancyclic if every vertex of G is contained in cycles of length $3, 4, \ldots, n$. Let m_n denote the minimum number of edges of a vertex pancyclic graph on n vertices. We show that $m_3 = 3$, $m_4 = 5$, $m_5 = 7$, $m_6 = 9$, and $\frac{3}{2}n < m_n \leqslant \lfloor \frac{5}{3}n \rfloor (n \geqslant 7)$.

1. Preliminaries

We use [1] for terminology and notation not defined here and consider simple graphs only. Let G be a graph of order $n \geqslant 3$. We say that G is *vertex pancyclic* if every vertex of G is contained in cycles of length $3, 4, \ldots, n$. We denote by m_n the minimum number of edges of a vertex pancyclic graph on n vertices ($n \geqslant 3$). In this note we give exact values of m_n for $3 \leqslant n \leqslant 6$, and we determine lower and upper bounds on m_n for $n \geqslant 7$.

Proposition 1. $m_3 = 3$, $m_4 = 5$, $m_5 = 7$, *and* $m_6 = 9$.

Proof. It is easy to check that the graphs of Figs. 1 and 2 are the vertex pancyclic graphs on 3, 4, 5, and 6 vertices with minimum size. □

Proposition 2. *The only 3-regular vertex pancyclic graphs are* K_4 *and* $K_2 \times K_3$ (*shown in Fig.* 2).

Proof. Consider a vertex v of a 3-regular vertex pancyclic graph G and its neighbors u_1, u_2, and u_3. Without loss of generality assume $u_1u_2 \in E(G)$. Clearly, $|V(G)| \geqslant 4$. Since v is on a cycle of length 4, there exists a vertex $w \in V(G) \backslash \{v\}$ such that w is adjacent to at least two vertices of $\{u_1, u_2, u_3\}$. Suppose $w \in \{u_1, u_2, u_3\}$. Then we may assume without loss of generality $u_2u_3 \in E(G)$. If $G \neq K_4$, we consider the neighbor

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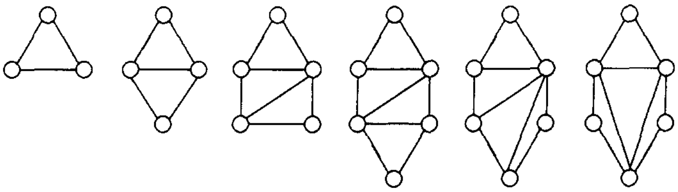

Fig. 1. The smallest vertex pancyclic graphs on $n \leqslant 6$ vertices.

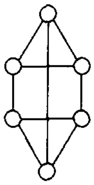

Fig. 2. $K_2 \times K_3$.

$x \notin \{v, u_2\}$ of u_1. If $xu_3 \in E(G)$, then x is a cut vertex of G, an obvious contradiction. Let $y \notin \{v, u_2\}$ denote the other neighbor of u_3. Since v is on a cycle of length 5, $xy \in E(G)$, and, since x and y are on cycles of length 3, there is a vertex $z \in N(x) \cap N(y)$. Now z is a cut vertex of G, a contradiction. Hence $w \notin \{u_1, u_2, u_3\}$. If w is adjacent to all vertices of $\{u_1, u_2, u_3\}$, we easily obtain a contradiction by observing that u_3 is a cut vertex of G. Without loss of generality now assume w is adjacent to u_1 and precisely one vertex of $\{u_2, u_3\}$. Suppose $wu_2 \in E(G)$ (hence $wu_3 \notin E(G)$). Consider the neighbor $x \notin \{u_1, u_2\}$ of w. Now x and u_3 play the same role as x and y above, and we obtain a contradiction. Hence $wu_3 \in E(G)$ (and $wu_2 \notin E(G)$). Since w is on a cycle of length 3, it is clear there exists a vertex $x \in N(w) \cap N(u_3)$. Using that u_2 and x are on cycles of length 4, it is clear that $u_2x \in E(G)$, hence $G = K_2 \times K_3$. □

In the next two sections we, respectively, obtain a lower and upper bound on m_n in case $n \geqslant 7$.

2. A lower bound

For $n \geqslant 7$, the next result gives a lower bound for the minimum size of a vertex pancyclic graph of order n.

Theorem 3. $m_n > \frac{3}{2}n$ ($n \geqslant 7$).

Proof. Let G be a vertex pancyclic graph of order $n \geqslant 7$. Then $\delta(G) \geqslant 2$. If $\delta(G) \geqslant 3$, then $m_n \geqslant \frac{3}{2}n$, and equality only holds if G is 3-regular, which is impossible by

Proposition 2. Thus for $\delta(G) \geqslant 3$ the result holds. Now assume $\delta(G) = 2$ and let T denote the set of vertices of G having degree 2. It is clear that no two vertices of T are adjacent (since every vertex is contained in cycles of length 3 and 4). Together with the fact that G is hamiltonian (contains a cycle of length n) this implies that $|V(G)\setminus T| \geqslant |T|$. It is also clear that the neighbors of a vertex of degree 2 are adjacent (since every vertex is on a cycle of length 3), and that two vertices of degree 2 do not have the same neighbor set (since G is hamiltonian). If $|V(G)\setminus T| \geqslant |T|$, this implies that every vertex of $V(G)\setminus T$ has degree at least 4, and, since every vertex lies on a cycle of length 5, we easily obtain that $\Delta(G) \geqslant 5$, hence $m_n > \frac{3}{2}n$. Henceforth assume $|V(G)\setminus T| > |T|$. Define the binary relation R on T as follows: uRv if and only if there exists a path P between u and v such that the vertices of P are alternately in T and $V(G)\setminus T$. Clearly, R is an equivalence relation and induces a partition π of T. Denote the number of one element classes of π by t_0, and denote the other classes of π (if any) by $S_1, S_2, \ldots, S_k$, where $k = |\pi| - t_0$. Set $t_i = |S_i|$ $(i = 1, 2, \ldots, k)$. Then $|T| = t_0 + t_1 + \cdots + t_k$ and at least one of t_0 and k is nonzero. We make the following observations.

(a) Consider a set of the partition with exactly one element $v \in T$. Clearly, v has two neighbors w_1 and w_2 in G and for all vertices $u \in T\setminus\{v\}$: $N(u) \cap N(v) = \emptyset$. Since v lies on cycles of length 4 and 5, it is easy to show that $d(w_1) + d(w_2) \geqslant 7$.

(b) Consider a set S_i with $t_i \geqslant 2$ elements $v_1, v_2, \ldots, v_{t_i} \in T$. Clearly v_j has two neighbors w_j and w_{j+1} in G $(j = 1, \ldots, t_i)$ and for all $r \in \{1, 2, \ldots, k\}\setminus\{i\}$: $N(S_i) \cap N(S_r) = \emptyset$. Since v_j lies on cycles of length 4 and 5 $(j = 1, \ldots, t_i)$ it is easily shown that $d(w_1) + \cdots + d(w_{t_i+1}) \geqslant 4(t_i + 1)$.

(c) The t_{k+1} vertices of $V(G)\setminus T$ that are nonadjacent to vertices of T have degree at least 3.

From (a), (b), and (c) it follows that $n = 3t_0 + t_{k+1} + 2(t_1 + \cdots + t_k) + k$, and $m \geqslant \frac{1}{2}\{2(t_0 + \cdots + t_k) + 3t_{k+1} + 7t_0 + 4(t_1 + \cdots + t_k) + 4k\} = \frac{3}{2}\{3t_0 + t_{k+1} + 2(t_1 + \cdots + t_k) + \frac{4}{3}k\}$. Hence $m > \frac{3}{2}n$, unless $k = 0$. In the latter case, all vertices contributing to t_{k+1} have degree 3, while the degree sum of the two neighbors of each vertex v in (a) above is equal to 7. It is not difficult to check that this is impossible if $n \geqslant 7$. $\square$

3. An upper bound

We proceed by giving a method to construct a vertex pancyclic graph of arbitrary order $n \geqslant 7$ with size $\lfloor \frac{5}{3}n \rfloor$.

Construction. Start with a cycle $v_1 v_2 \ldots v_n$ of length $n \geqslant 7$ and add the edges $v_n v_{n-2}$, $v_n v_{n-4}$ and $v_{n-2} v_{n-4}$. Now distinguish two cases.

I. $n < 10$. Add the edge $v_n v_{n-5}$. If $n \geqslant 8$, add the edge $v_{n-5} v_1$. If $n \geqslant 9$, also add the edge $v_{n-6} v_1$.

II. $n \geqslant 10$. For all j with $1 \leqslant j \leqslant \frac{1}{3}(n-10)$ add the edges $v_j v_{n-4-2j}$ and $v_j v_{n-3-2j}$. Distinguish three subcases.

II.1. $n = 3k-2$, $k \geqslant 4$. Add the edges $v_{k-3}v_{k-1}$, $v_{k-3}v_{k+1}$ and $v_{k-1}v_{k+1}$.

II.2. $n = 3k-1$, $k \geqslant 4$. Add the edges $v_{k-3}v_{n+4-2k}$, $v_{k-2}v_k$, $v_{k-2}v_{k+2}$ and $v_k v_{k+2}$.

II.3. $n = 3k$, $k \geqslant 4$. Add the edges $v_{k-3}v_{n+4-2k}$, $v_{k-3}v_{n+3-2k}$, $v_{k-2}v_k$, $v_{k-2}v_{k+2}$ and $v_k v_{k+2}$.

We leave it to the reader to check that in all cases the constructed graphs are vertex pancyclic and have size $m = \lfloor \frac{5}{3}n \rfloor$. More specifically, for $k \geqslant 3$, if $n = 3k-2$, then $m = \frac{5}{3}n - \frac{2}{3}$, if $n = 3k-1$, then $m = \frac{5}{3}n - \frac{1}{3}$, and if $n = 3k$, then $m = \frac{5}{3}n$.

References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Macmillan, London and Elsevier, New York, 1976).